Extending the Natural Log

We defined

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

for $x > 0$.

Recall the domain of $\ln(x)$ is all positive real numbers.

If $x < 0$, then $\ln(|x|)$ makes sense. By the chain rule,

$$\frac{d}{dx}(\ln(|x|))$$

$$= \frac{d}{dx}(\ln(-x))$$

$x < 0$, so $-x > 0$

$$= \frac{1}{-x} \cdot \frac{d}{dx}(-x)$$

$$= -\frac{1}{x} \cdot (-1) = \frac{1}{x}$$ ✓

So

$$\boxed{\frac{d}{dx}(\ln(|x|)) = \frac{1}{x}}$$

for all $x \neq 0$.

Example 1: (another hard derivative)

Let $0 < x < \frac{\pi}{2}$.

Let $f(x) = \left((\cos(x))^{\sin(x)} \right)^{x}$.

Find $f'(x)$.

$$f(x) = (\cos(x))^{x\sin(x)}$$

by rules of exponents.

Write

$$f(x) = e^{\ln(f(x))}$$

$$= e^{\ln\left((\cos(x))^{x\sin(x)}\right)}$$

$$= e^{x\sin(x)\ln(\cos(x))}$$

$$f'(x) = e^{x\sin(x)\ln(\cos(x))} \cdot \frac{d}{dx}\left(x\sin(x)\ln(\cos(x))\right)$$

$$f'(x) = e^{x\sin(x)\ln(\cos(x))}$$

$$\cdot\ \underbrace{\left(x \underbrace{\frac{d}{dx}\left(\sin(x)\ln(\cos(x))\right)}_{\text{another product rule}} + \sin(x)\ln(\cos(x))\right)}_{\text{product rule}}$$

$$= \sin(x)\underbrace{\frac{d}{dx}\left(\ln(\cos(x))\right)}_{\text{chain rule}} + \ln(\cos(x))\cos(x)$$

$$= \sin(x)\cdot\frac{1}{\cos(x)}\left(-\sin(x)\right) + \ln(\cos(x))\cos(x)$$

$$= -\sin(x)\tan(x) + \ln(\cos(x))\cos(x)$$

Then

$$f'(x) = e^{x\sin(x)\ln(\cos(x))} \cdot \left(\sin(x)\ln(\cos(x)) + x \cdot \left(-\sin(x)\tan(x) + \ln(\cos(x))\cos(x) \right) \right)$$

$$\left(= (\cos(x))^{x\sin(x)}, \text{ if you like} \right)$$

Given $f(x) = (g(x))^{h(x)}$

(assuming this makes sense, $g > 0$,

and h, g are differentiable),

$$\frac{d}{dx}\left(f(x)\right) = f(x) \cdot \frac{d}{dx}\left(h(x)\ln(g(x))\right)$$

$$= \boxed{f(x)\left(h'(x)\ln(g(x)) + \frac{h(x)\,g'(x)}{g(x)}\right)}$$

The technique we have used to deal with the derivative of $(g(x))^{h(x)}$ is called

logarithmic differentiation.

Example 2: (more logarithmic differentiation)

$$f(x) = \sin(x)\cos(2x)\,e^{x}\,(x^2+1)^{50}\tan(4x).$$

Find $f'(x)$.

Observe that

$$\ln(f(x)) = \ln(\sin(x)) + \ln(\cos(2x)) + \ln(e^{x}) + \ln((x^2+1)^{50}) + \ln(\tan(4x))$$

We can simplify this to

$$\ln(\sin(x)) + \ln(\cos(2x)) + x\ln(e) + 50\ln(x^2+1) + \ln(\tan(4x)).$$

Since $\ln(e) = 1$, we finally have

$$\ln(\sin(x)) + \ln(\cos(2x)) + x + 50\ln(x^2+1) + \ln(\tan(4x)).$$

Then differentiate!

Or ask Wolfram Alpha...

Logs to different bases

Let "a" be a real number,

$a > 0$, $a \neq 1$.

We <u>define</u>

$$\boxed{\log_a(x) = \frac{\ln(x)}{\ln(a)}}.$$

We don't want $a = 1$ since

$\ln(1) = 0$.

$$\frac{d}{dx}\left(\log_a(x)\right) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right)$$

$$= \frac{1}{\ln(a)}\frac{d}{dx}\left(\ln(x)\right)$$

$$= \boxed{\frac{1}{x\ln(a)}}.$$

Observe $\log_a(a) = \frac{\ln(a)}{\ln(a)} = 1.$

We'll look at graphs later.

Exponentials to Different Bases

Let $a > 0$, $a \neq 1$.

Define $a^x = e^{x\ln(a)}$

for all x.

Then $\log_a(a^x)$

$$= \log_a\left(e^{x\ln(a)}\right)$$

$$= x\ln(a)\log_a(e)$$

$$= x\cancel{\ln(a)}\,\frac{\overbrace{\ln(e)}^{=1}}{\cancel{\ln(a)}}$$

$$= x$$

This shows $\log_a(x)$ and a^x are inverse functions.

Furthermore,

$$\frac{d}{dx}\left(a^x\right) = \frac{d}{dx}\underbrace{\left(e^{x\ln(a)}\right)}_{\text{chain rule}}$$

$$= e^{x\ln(a)} \cdot \ln(a)$$

$$= \boxed{a^x \ln(a)}.$$

Note if $a=e$, $\ln(e)=1$, so it disappears in the formula.

Working backwards,

$$\boxed{\int a^x\,dx = \frac{a^x}{\ln(a)} + C}$$

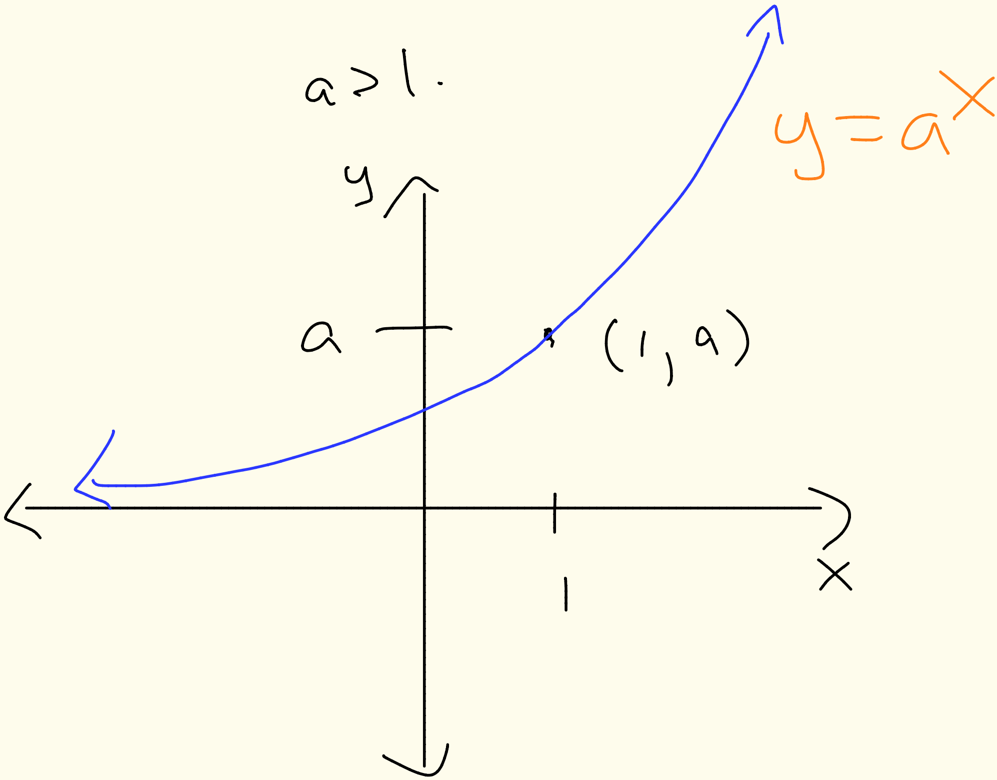
Very soon, we'll figure out

$$\int \log_a(x)\,dx \; -$$

but not today!

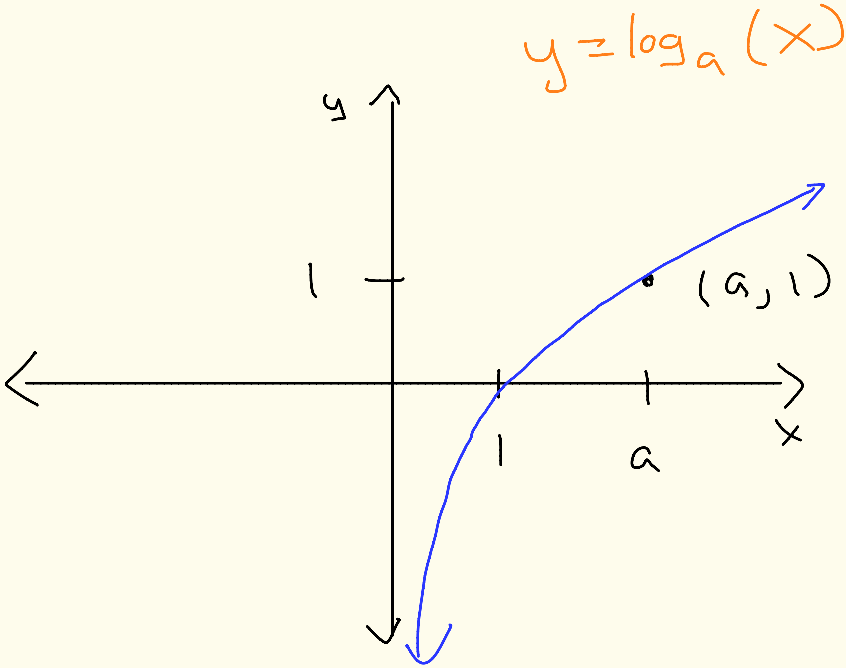
Graphs

$a > 1.$



$\lim_{x \to -\infty} a^x = 0, \quad \lim_{x \to \infty} a^x = \infty$

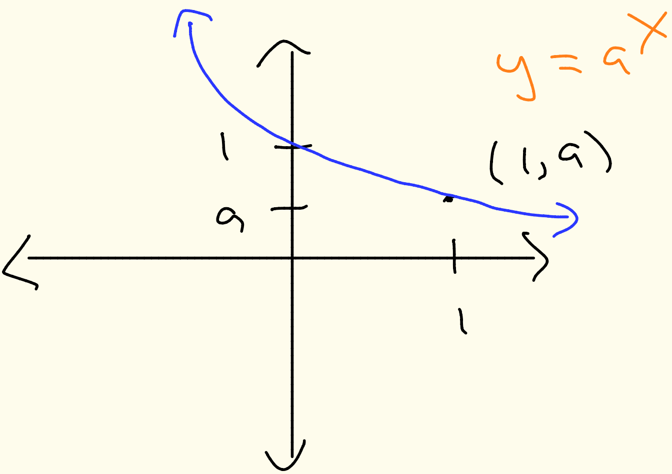
concave up, always increasing

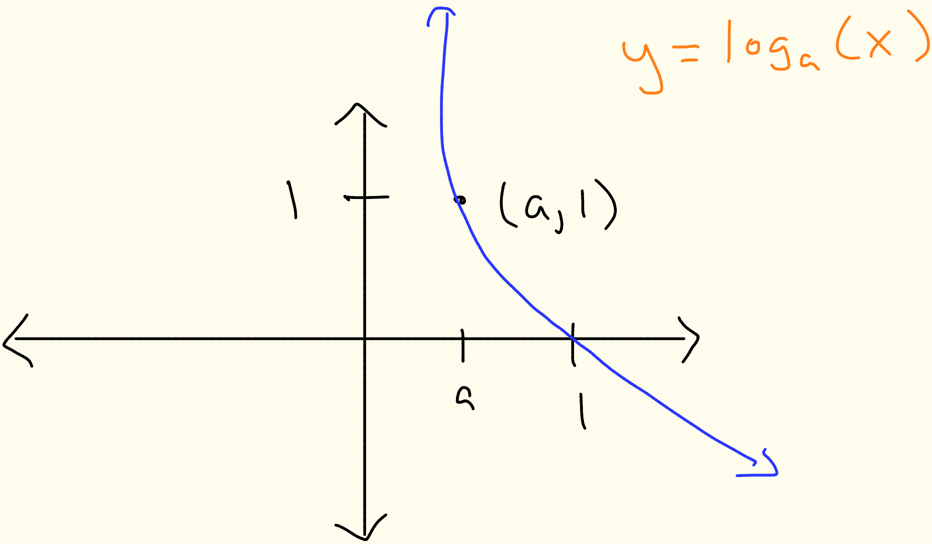


$$\lim_{x\to\infty} \log_a(x) = \infty, \quad \lim_{x\to 0^+} \log_a(x) = -\infty$$

Concave down, always increasing

$0 < a < 1$





Example 3: Find $\int \log_5(9^x)\,dx$.

$$\int \log_5(9^x)\,dx = \int \frac{\ln(9^x)}{\ln(5)}\,dx \quad \text{(def.)}$$

$$= \frac{1}{\ln(5)} \int \ln(9^x)\,dx$$

$$= \frac{1}{\ln(5)} \int x\ln(9)\,dx$$

(log rules)

$$= \frac{\ln(9)}{\ln(5)} \int x\,dx$$

$$= \boxed{\frac{\ln(9)}{\ln(5)}\left(\frac{x^2}{2}+C\right)}$$

Example 4: Compute $\int_0^1 7e^{x^4}x^3\,dx$.

$$u = x^4$$

$$du = 4x^3\,dx, \text{ so } \frac{du}{4} = x^3\,dx.$$

When $x=0$, $u=0$, and when $x=1$, $u=1$. The integral becomes

$$\int_0^1 7 \cdot e^u \cdot \frac{du}{4} = \frac{7}{4}\int_0^1 e^u\,du$$

$$= \frac{7}{4}\left(e^u \Big|_0^1\right)$$

$$= \boxed{\frac{7}{4}(e-1)}$$

Example 5: Evaluate $\int 4^x(2^x-5)\,dx$.

First, does $a^x b^x = (ab)^x$?

Well, $\ln(a^x b^x) = \ln(a^x) + \ln(b^x)$

$$= x(\ln(a) + \ln(b))$$

$$= x\ln(ab)$$

$$= \ln((ab)^x),$$

so exponentiating,

$$a^x b^x = (ab)^x \checkmark$$

Then

$$\int 4^x (2^x - 5)\, dx$$

$$= \int (4^x 2^x - 4^x \cdot 5)\, dx$$

$$= \int (8^x - 4^x \cdot 5)\, dx$$

$$= \boxed{\frac{8^x}{\ln(8)} - 5 \cdot \frac{4^x}{\ln(4)} + C}$$